Primordial gravity waves fossils and their use in testing inflation

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(Dated: August 25, 2010)

A new effect is described by which primordial gravity waves leave a permanent signature in the large scale structure of the Universe. The effect occurs at second order in perturbation theory and is sensitive to the order in which perturbations on different scales are generated. We derive general forecasts for the detectability of the effect with future experiments, and consider observations of the pre-reionization gas through the 21 cm line. It is found that the Square Kilometre Array will not be competitive with current cosmic microwave background constraints on primordial gravity waves from inflation. However, a more futuristic experiment could, through this effect, provide the highest ultimate sensitivity to tensor modes and possibly even measure the tensor spectral index. It is thus a potentially quantitative probe of the inflationary paradigm.

Introduction.— It has been proposed that redshifted 21 cm radiation, from the spin flip transition in neutral hydrogen, might be a powerful probe of the early universe. The era before the first luminous objects reionized the universe–around redshift 10–contains most of the observable volume of the universe, and 21 cm radiation is the only known probe of these so called dark ages (see Furlanetto et al. [1] for a review). The density of the hydrogen could be mapped in 3D analogous to how the cosmic microwave background (CMB) is mapped in 2D. The wealth of obtainable statistical information may allow for the detection of many subtle effects which could probe the early universe. In particular, the primordial gravity wave background, also referred to as tensor perturbations, are of considerable cosmological interest.

Inflation robustly predicts the production of tensor perturbations with a nearly scale-free spectrum, however, their amplitude is essentially unconstrained theoretically. The amplitude of the tensor power spectrum is quantified by r, the tensor to scalar ratio. The current upper limit is $r < 0.24$ at 95% confidence [2], however upcoming CMB measurements will be sensitive down to r of a few percent [3]. The current limits on r correspond to characteristic primordial shear on the order of 10^{-5} per logarithmic interval of wavenumber.

Several probes of gravity waves using the pre-reionization 21 cm signal have been proposed. These include polarization [4] and redshift space distortions [5]. Dodelson et al. [6] considered the weak lensing signature of gravity waves and found that the signal is sensitive to the so called metric shear. This is closely related to the present work.

Here we describe a mechanism by which primordial gravitational waves may leave an imprint in the statistics of the large scale structure (LSS) of the universe. This signature becomes observable when the gravity wave enters the horizon and begins to decay.

Mechanism.— In the following, Greek indices run from 0 to 3 and lower case Latins from 1 to 3. Latin indices are always raised and lowered with Kronecker deltas. Commas denote partial derivatives, and an over-dot ($\dot{\#}$) represents a derivative with respect to the cosmological conformal time. Finally, we adopt a mostly positive metric signature $(-1, 1, 1, 1)$.

We start with an inflating universe with some distribution of previously generated tensor modes that are now super horizon. Scalar, vector and smaller scale tensor modes may exist but their contribution to the metric is ignored. The line element is given by

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]. \qquad (1)$$

where a is the scale factor, η the conformal time and a spatially flat background geometry has been assumed. The metric perturbations h_{ij} are assumed to be transverse and traceless and thus contain only tensor modes. The elements of h_{ij} are also assumed to be small such that only leading order terms need be retained. The assumption that all tensor modes under consideration are super horizon implies that $k_h \ll \dot{a}/a$, where k_h denotes the wave numbers of tensor modes. The frame in which the line element takes the form in Eq. 1 will hereafter be referred to as the cosmological frame (CF).

By the equivalence principle, it is possible to perform a coordinate transformation such that the space-time appears locally Minkowski at a point. New coordinates are defined in which the tensor modes are gauged away at the origin:

$$\tilde{x}^\alpha = (x^\alpha + \frac{1}{2}h^\alpha{}_\beta x^\beta), \qquad (2)$$

where the elements $h_{0\alpha}$ are taken to be zero. The metric now takes the form (up to first order in h_{ij})

$$ds^2 = a^2 \left[-d\eta^2 + \delta_{ij}d\tilde{x}^i d\tilde{x}^j - \tilde{x}^c \partial_\alpha h_{\beta c} d\tilde{x}^\alpha d\tilde{x}^\beta\right]. \qquad (3)$$

This frame will be loosely referred to as the locally Friedmann frame (LFF), because in these coordinates the metric is locally that of an unperturbed FLRW Universe. We will give quantities in these coordinates a tilde ($\tilde{\#}$) to distinguish them from their counterparts in the CF. It

is seen from Eq. 3 that the local effects of gravity waves are suppressed not only by the smallness of h_{ij} but also by k_h/k where $k = L^{-1}$ and L is some length scale of interest. This will be important in justifying some later assumptions. Note that for super horizon gravity waves, temporal derivatives are much smaller than spacial ones.

On small scales, inflation generates scalar perturbations which are then carried to larger scales by the expansion. By the equivalence principle, physical processes on small scales can not know about the long wavelength tensor modes. As such these small scale scalar modes must be uncorrelated with the long wavelength tensor modes. We assume statistical homogeneity and isotropy in the LFF as would be expected from inflation. The power spectrum of scalar perturbations can then be written as a function of only the magnitude of the wave number, i.e., $\tilde{P}(\tilde{k}_a) = \tilde{P}(\tilde{k})$. This applies only within the local patch near the point where the tensor mode was gaged away. The average in the definition of the scalar power spectrum is over realizations of the scalar map, but not the tensor map.

In the CF, the isotropy is broken. Transforming back to cosmological coordinates maps $\tilde{k}_i \to k_i - k_j {h_i}^j/2$. The power spectrum becomes sheared:

$$P(k_a) = \tilde{P}(k) - \frac{k_i k_j h^{ij}}{2k}\frac{d\tilde{P}}{dk} + O(\frac{k_h}{k}h_{ij}) + O({h_{ij}}^2). \quad (4)$$

If the metric perturbations are not assumed to be traceless, the right hand side of this equation gains an additional term proportional to this trace. This deviation from isotropy is not observable since any possible observation would take place in the LFF.

It is noted that the leading order correction to CF power spectrum is not suppressed by k_h/k. It is therefore not expected that the residual terms in the LFF metric (Eq. 3) can break isotropy to undo CF anisotropy. However it was the CF in which the power spectrum should be isotropic, then there would be *observable* anisotropy in the LFF. This would be a violation of the equivalence principle, since an experiment local in both space and time would be able to detect the super horizon tensor modes by measuring the power spectrum of the locally generated scalar perturbations.

We would now like to evolve the system to some later time when observations can be made. Ignoring the internal dynamics of the scalar perturbations, we solve for their evolution as if they were embedded in a sea of test particles. This is trivial since an object at coordinate rest in the CF will remain at rest for any time dependence of h_{ij} (this is true at all orders). At some point well after inflation, when the universe is in its deceleration phase, the horizon will become larger than the length scale of the tensor modes. The tensor modes will then decay by redshifting, and after some period of time the metric perturbations h_{ij} become negligible. The CF and LFF then become equivalent and both correspond to the frame in which observations can be made. The distribution of test particles is the same as it initially was in the CF. As such, the initially physically isotropic power spectrum now contains a measurable local anisotropy given by Eq. 4. The values of the initial metric perturbations can be determined by measuring this distortion at any time in the future, constituting a fossil of the initial tensor modes.

The scalar perturbations remain Gaussian but become non-stationary, and the trispectrum gains the corresponding terms. This is analogous to the apparent distortions expected in the CMB and 21 cm fields induced by gravitational lensing. Similarly the bispectra of mixed scalars and tensors were calculated in Maldacena [7], employing similar methodology to that presented here.

The effect described here is a second order perturbation theory effect, in that it is a small effect due to tensor modes on the already small scalar perturbations. This coupling occurs in the initial conditions, not between the dynamics of the scalars and tensors. The simple argument presented above avoided the complication of a full second order calculation, but it is expected that such calculations would yield the same results. Specifically, an expression agreeing with Eq. 4, to relevant order, was derived in Giddings and Sloth [8, Eq. 4.5] as part of a longer calculation.

Tests of inflation.— The above arguments relied on perturbations on large scales being generated before perturbations on small scales. This is the case in any conceivable model of inflation, however it is not be the case in all scenarios. As an illustrative example, in the cosmic defect scenario perturbations are generated on small scales and then causally transported to larger scales as the universe evolves. It is argued that in this scenario, tensor perturbations leave no fossils. A detection of primordial tensors by another means (CMB B-modes for example) with an observed lack of the corresponding fossils would provide a serious challenge to inflation.

The most specific prediction of single field inflation is the power spectrum of tensor modes, defined by

$$(2\pi)^3\delta(k_a - k'_a)P_h(k_a) \equiv \langle h_{ij}(k_a)h^{ij}(k'_a)\rangle. \quad (5)$$

Given the amplitude of the scalar power spectrum A_s, the tensor power spectrum is fixed by a single parameter, the tensor to scalar ratio r. The shape of the spectrum is then nearly scale-free:

$$P_h = \frac{2\pi^2 r A_s}{k^3}\left(\frac{k}{k_0}\right)^{n_t}. \quad (6)$$

We follow the WMAP conventions for defining P_h, A_s and r [9]. The spectral index fixed by the consistency relation, $n_t = -r/8$ [10]. The pivot scale is taken to be $k_0 = 0.002\,\mathrm{Mpc}^{-1}$ and we assume the WMAP7 central value for A_s of 2.46×10^{-9}.

Because r is likely small, any deviation from a scale-free spectrum will be difficult to measure, making the

verification of the consistency relation correspondingly difficult. The CMB is sensitive primarily to large scale tensor modes, with smaller scale modes having decayed by recombination. Cosmic variance and lensing contamination will likely prevent a measurement of n_t from the CMB, unless the lensing can be cleaned from the signal [11]. Conversely, the amplitude of the fossil signal does not decay as the universe expands. It may thus be possible to make a measurement of the spectral index, provided r is sufficiently large.

Statistical detection in LSS.— In practice, the tensor gravity wave fossils could be reconstructed by applying quadratic estimators to the density field. Aside from the increased dimensionality, this is identical to the manner in which lensing shear is reconstructed [12, 13]. Rather than considering the statistics of such estimators, here we follow a simpler line of reasoning to approximate the accuracy to which the tensor parameter can be measured.

We begin by asking how well a long wavelength, tensor mode can be reconstructed from its effects on the scalar power spectrum (Eq. 4). The metric perturbations are assumed to be spatially constant and take the form

$$h_{ij} = h_+ e^+_{ij}(\hat{z}) + h_\times e^\times_{ij}(\hat{z}) \tag{7}$$

where e^+_{ij} and $e^\times_{ij}$ are the polarization tensors and the $\hat{z}$ direction of propagation is chosen for convenience. The uncertainty on the scalar power spectrum is

$$[\Delta P(k_a)]^2 = 2\,[P(k_a) + N]^2\,, \tag{8}$$

where N is the noise power. We use a Fisher Matrix analysis to sum this information over all k_a to determine the corresponding uncertainty on the shear h_+ and $h_\times$. Assuming an experiment whose noise is sub dominant to sample variance ($N \ll P$), the resulting variance is inversely proportional to the number of modes surveyed:

$$\left(\Delta h^C\right)^2 \sim \left[V(k_{max}/2\pi)^3\right]^{-1}, \tag{9}$$

where h stands for either h_+ or $h_\times$ (the superscript C indicates that the formula applies for spatially constant h), V is the volume of the survey and k_{max} is set by the resolution of the survey. The constant of proportionality depends on the shape of the unsheared power spectrum $\tilde{P}(k)$, but to within a few tens of percent it is unity. 21 cm emission will be difficult to observe on large scales [1], however it is small scales that dominate the number of modes and thus the reconstruction. It is only the coherence of small scale anisotropy that must be measured on large scales.

Given the reconstruction uncertainty on a spatially constant shear, and the fact that reconstruction noise is scale independent (white) [12], the noise power spectrum for spatially varying tensor modes is then

$$N_h = 4V(\Delta h^C)^2 = 4\left(\frac{2\pi}{k_{max}}\right)^3. \tag{10}$$

The factor of four comes from the definition of the power spectrum in Eq. 5, noting that $\langle h_{ij}h^{ij}\rangle = 4\langle h^2\rangle$.

We now sum over k_a [1] to determine the signal to noise as a function of tensor power spectrum amplitude r. The signal to noise ratio squared is then

$$SNR^2 = \sum_{k_a,\{+,\times\}} \frac{P_h^2}{2(N_h+P_h)^2} \tag{11}$$

$$\approx V\int_{k_{lower}}^{k_{upper}} \frac{dk\,k^2}{2\pi^2}\frac{P_h^2(k)}{(N_h+P_h)^2}. \tag{12}$$

It is seen from the redness of the spectrum P_h (Eq. 6) that the result is completely independent of the upper limit of integration. The same redness makes the final result extremely sensitive to the lower limit. As described above, the fossil of a primordial tensor mode can only be observed once the mode has decayed. This begins to happen when the scale of the gravity wave becomes comparable to the horizon scale, and as such, the largest scale observable mode has wavelength $k_{lower} \approx aH$.

For an initial detection, we assume that noise dominates sample variance at each k_a, i.e., $N_h \gg P_h$. Setting the signal to noise ratio to be 2, for a 95% confidence detection, yields a minimum detectable amplitude of

$$r_{min} = \frac{32\pi^2}{A_s k_{max}^3}\left(\frac{6}{VV_H(z)}\right)^{1/2} \tag{13}$$

where $V_H \equiv (aH)^{-3}$.

While the observability of 21 cm radiation depends on the reionizaton model, one regime in which a strong signal may exist is near redshift 15 [1]. The planned Square Kilometer Array (SKA) will aim to probe this era with 10 km baselines [14]. Assuming a survey volume of $200\,(\mathrm{Gpc}/h)^3$ and a noiseless measurement, the limit on r achievable with SKA will be

$$r_{min} \approx 7.3\left(\frac{1.2\,\mathrm{Mpc}/h}{k_{max}}\right)^3\left[\frac{200\,(\mathrm{Gpc}/h)^3}{V}\frac{3.3\,(\mathrm{Gpc}/h)^3}{V_H}\right]^{1/2}. \tag{14}$$

While this constraint is not competitive with current constraints from the CMB, it is a strong function of the resolution of the experiment. The Low Frequency Array (LOFAR) for instance, has baselines extending to 400 km. However LOFAR will not have the sensitivity to probe the dark ages [15]. It is the physical shear due to gravity waves at the source that is being measured, and all light propagation effects, such as the lensing considered in Dodelson et al. [6], have been ignored.

[1] From this point forward, k_a will refer to the wave number of a tensor mode, not a scalar mode. The exception will be k_{max} which is the smallest scale at which a scalar can be resolved.

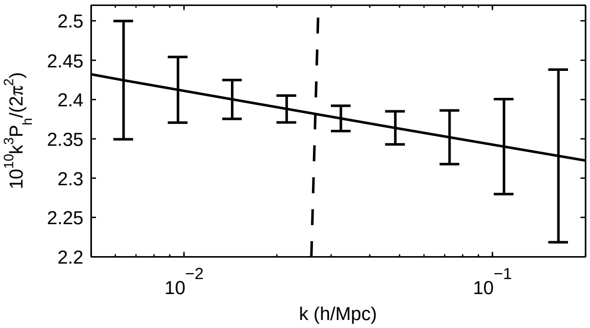

FIG. 1: Primordial tensor power spectrum obeying the consistency relation for $r = 0.1$. The solid line is the tensor power spectrum. Error bars represent the reconstruction uncertainty on the binned power spectrum for a perfect experiment, surveying $200\,(\mathrm{Gpc}/h)^3$ and resolving scalar modes down to $k_{max} = 168h/\mathrm{Mpc}$. The dashed, nearly vertical, line is the reconstruction noise power. The non-zero slope of the solid line is the deviation from scale-free.

Similar arguments are used to find the achievable error on the spectral index n_t. Properly considering the degeneracy with r, the error on n_t is:

$$\Delta n_t = F\left[\left(\frac{2\pi}{k_{max}}\right)^3 \frac{1}{rA_sV}\right]^{1/2}, \qquad (15)$$

where F is a function of the combination of parameters $V_H/(k_{max}^3 rA_s)$. In the limit that $P_h(k = aH) \gg N_h$, which is the limit in which a measurement of n_t is possible, F is approximately 6. Assuming the same volume and redshift as above, and that $r = 0.1$, the consistency relation is tested at the 2 sigma level for $k_{max} = 168h/\mathrm{Mpc}$. The tensor power spectrum and error bars for this scenario are shown in Fig. 1.

Such a measurement is very futuristic indeed, requiring a nearly filled array with greater than thousand kilometre baselines. Note that such an experiment would be sensitive to r down to the 10^{-6} level. Also, higher redshifts contain even more information, though their observation is technically more challenging.

Discussion.— Aside from the technical challenge of mapping the 21 cm signal over hundreds of cubic gigaparsecs and down to scales smaller than a megaparsec, there may be other competing effects that could hinder a detection. Of primary concern is weak lensing which also shears observed structures, creating apparent local anisotropies. The weak lensing shear is of order a few percent, and is thus many orders of magnitude greater than gravity wave shear. However, the 3D map of gravity wave shear will be transverse, transforming intrinsically as a tensor. To linear order, the lensing pattern is the gradient of a scalar. Even at higher order, lensing always maps one point in space to another and is thus at most vector like. This test does not exist for the CMB or lensing due to the lower dimensionality of these probes.

Also of concern is the preservation of the anisotropy on small scales. The scale corresponding to $k = 168h/\mathrm{Mpc}$ is still larger than the Jeans length at these redshifts, and as such hydrogen should trace the dark matter. However, the evolution of scalar perturbations is mildly nonlinear, and it is possible that this evolution will erase the anisotropy. Detailed analysis of the nonlinear erasure of the anisotropy is deferred to future investigation.

There has been much recent interest in searching for anisotropy, and this has some implications for the fossil signal. The constraints on quadrupolar isotropy in LSS by Pullen and Hirata [16] should already imply a weak constraint at the $r \lesssim 10^6$ level. Constraints from the CMB are not relevant however, since modes spanning the surface of last scatter remain super horizon today.

CMB B-modes will be the most sensitive probe of primordial gravity waves in the next generation of experiments. However, fossils may eventually be sensitive well below the limits of the CMB.

We would like to thank Patrick McDonald, Latham Boyle, Adrian Erickcek, Neil Barnaby, Neal Dalal, Chris Hirata and Eiichiro Komatsu for helpful discussions. KM is supported by NSERC Canada.

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